The Quasi-exact models in two-dimensional curved space based on the generalized CRS Harmonic Oscillator

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Abstract

In this paper, by searching the relation between the radial part of Higgs harmonic oscillator in the two-dimensional curved space and the generalized CRS harmonic oscillator model, we can find a series of quasi-exact models in two-dimensional curved space based on this relation.

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I. INTRODUCTION

The quasi-exactly solvable quantum problems was a remarkable discovery in last century[1]. This kind of problem can be solved by Lie algebra[2] or the analytical method[3]. Meanwhile, the quantum nonlinear harmonic oscillator (QNHO) has been studied with great interest [6–12]. Wang and Liu[6] generalized a class of QNHO which is called CRS model[7, 8] by the factorization method , whose Hamiltonian reads

$$H' = \epsilon \left(-\mathcal{K} \frac{d^2}{dx^2} - \lambda_{\mathrm{Q}} x \frac{d}{dx} \right) + V'(x), \quad (\epsilon = \frac{\hbar^2}{2m}), \tag{1}$$

where $\mathcal{K} = 1 + \lambda_{\mathrm{Q}} x^2$, λ_{Q} is a real number, m is the mass for the particle and

$$V'(x) = \epsilon \frac{(\beta X + \gamma)^2 + (\beta X + \gamma)(AX + B)}{\mathcal{K}(\frac{dX}{dx})^2} + C, \tag{2}$$

where β, γ and C are arbitrary real numbers; $X = X(x)$ is a function which is analytic nearby $x = 0$ here; the parameters A and B need to satisfy the equation

$$\mathcal{K} \frac{d^2 X}{dx^2} + \lambda_{\mathrm{Q}} x \frac{dX}{dx} = AX + B. \tag{3}$$

It is easily proved that the solutions of Hamiltonian (1) can be solved exactly with the potential (2) by the factorization method.

On the other hand, Higgs [4] and Leemon [5] introduced a generalization of the hydrogen atom and isotropic harmonic oscillator in a space with constant curvature. On 2-dimensional curved sphere, the Hamiltonian can be written as

$$H = \frac{1}{2m} \left(\pi^2 + \lambda_{\mathrm{G}} L^2 \right) + \mathcal{V}(r), \tag{4}$$

where $\pi = \mathbf{p} + \frac{1}{2}\lambda_{\mathrm{G}}[\mathbf{x}(\mathbf{x}\cdot\mathbf{p}) + (\mathbf{p}\cdot\mathbf{x})\mathbf{x}]$, $L^2 = \frac{1}{2} L_{ij} L_{ij}$, $r = \left|\sqrt{\mathbf{x}^2}\right|$ and λ_{G} is the curvature of the 2-dimensional curved sphere.

In this work, by studying the relation between the generalized CRS harmonic oscillator model[6] and the radial part of Higgs harmonic oscillator[4] in the two-dimensional curved space, we can find a series of quasi-exact models in two-dimensional curved space based on this relation. The paper is organized as follows. In Sec. 2, the link between a special

generalized CRS model and the Higgs model will be given; in Sec. 3, the generalized Higgs models which are quasi-exactly solvable will be shown; in Sec. 4, there will be a conclusion finally.

II. THE RELATION BETWEEN THE GENERALIZED CRS HARMONIC OSCILLATOR AND THE RADIAL PART OF HIGGS OSCILLATOR

A. The exactly solvable Higgs oscillator

Considering the two-dimensional Hamiltonian (4), we substitute it into the stationary Schrödinger equation, which $\mathcal{V}(r) = \frac{1}{2}m\omega^2 r^2$. The partial differential equation can be written as

$$\begin{aligned}&-\frac{\hbar^2}{2m}\left[(1+\lambda_G r^2)^2\frac{\partial^2}{\partial r^2} + \frac{(1+\lambda_G r^2)(1+5\lambda_G r^2)}{r}\frac{\partial}{\partial r} + \left(3\lambda_G + \frac{15\lambda_G^2 r^2}{4}\right) + (\lambda_G + \frac{1}{r^2})\frac{\partial^2}{\partial\theta^2}\right]\Psi(r,\theta)\\ &= (E_G - \frac{1}{2}m\omega^2 r^2)\Psi(r,\theta).\end{aligned} \tag{5}$$

which E_G is the stationary energy eigenvalue. If we make $\Psi(r,\theta) = e^{i\mathbb{m}_G\theta}\psi(r)$ and $\mathbb{m}_G$ is the angular parameter, it gives the radial part of above equation

$$\begin{aligned}&-\frac{\hbar^2}{2m}\left[(1+\lambda_G r^2)^2\frac{d^2}{dr^2} + \frac{(1+\lambda_G r^2)(1+5\lambda_G r^2)}{r}\frac{d}{dr} + \left(3\lambda_G - \lambda_G\mathbb{m}_G^2 + \frac{15}{4}\lambda_G^2 r^2 - \frac{\mathbb{m}_G^2}{r^2}\right)\right]\psi(r)\\ &= (E_G - \frac{1}{2}m\omega^2 r^2)\psi(r).\end{aligned} \tag{6}$$

Considering the work of Higgs [4], we know that the harmonic oscillator (4) on the 2-dimensional curve sphere with constant curvature λ_G has the radial wave function

$$\psi(r)_{N,\mathbb{m}_G} = r^{|\mathbb{m}_G|}\left(\frac{1}{1+\lambda_G r^2}\right)^{\frac{|\mathbb{m}_G|+2}{2}+\frac{m\omega'_G}{2\hbar\lambda_G}} F(-N, N+|\mathbb{m}_G|+1+\frac{m\omega'_G}{\lambda_G\hbar}, |\mathbb{m}_G|+1; \frac{\lambda_G r^2}{1+\lambda_G r^2}) \tag{7}$$

and the energy spectrum

$$E_{G(N,\mathbb{m}_G)} = \hbar\omega'_G(2N+|\mathbb{m}_G|+1) + \frac{\lambda_G\hbar^2}{2m}(2N+|\mathbb{m}_G|+1)^2, \tag{8}$$

which $\omega'_G = \sqrt{\omega^2 + \frac{\hbar^2\lambda_G^2}{4m^2}}$, N and $\mathbb{m}_G$ are both integer number here.

B. The exactly solvable generalized CRS harmonic oscillator

For the generalized CRS model[6], if we set the function and parameters in the potential (2) as $X(x) = \cos(2\Theta(x))$, $\beta = 2\lambda_Q(\mathbb{m}_Q+1)+\sqrt{\lambda_Q^2 + \frac{4m^2\omega^2}{\hbar^2}}$, $\gamma = 2\lambda_Q\mathbb{m}_Q-\sqrt{\lambda_Q^2 + \frac{4m^2\omega^2}{\hbar^2}}$, $A = -4\lambda_Q$, $B = 0$ and $C = \epsilon\left(\lambda_Q(\mathbb{m}_Q^2 - 1) + \mathbb{m}_Q\sqrt{\lambda_Q^2 + \frac{4m^2\omega^2}{\hbar^2}}\right)$, we get

$$V'(x) = \frac{1}{2}m\omega^2\left(\frac{\tan(\Theta(x))}{\sqrt{\lambda_Q}}\right)^2 - \frac{\lambda_Q\hbar^2}{8m}\left(1 + (1 - 4\mathbb{m}_Q^2)\csc^2(\Theta(x))\right), \tag{9}$$

where $\Theta(x) = \text{arcsinh}(\sqrt{\lambda_Q}x)$ and $\mathbb{m}_Q$ is a real number. With the potential above, by solving the generalized CRS eigen-equation

$$\left[\epsilon\left(-\mathcal{K}\frac{d^2}{dx^2} - \lambda_Q x\frac{d}{dx}\right) + V'(x)\right]\phi(x) = E_Q\phi(x), \tag{10}$$

we have the wavefunction

$$\begin{aligned}\phi(x) \;&=\; (-\sin^2(2\Theta(x)))^{-\frac{3}{4}}\sin^2(\Theta(x))\left(\frac{\tan(\Theta(x))}{\sqrt{\lambda_Q}}\right)^{|\mathbb{m}_Q|} \qquad (11)\\ &(\cos(\Theta(x)))^{\frac{|\mathbb{m}_Q|+2}{2}+\frac{m\omega'_Q}{2\hbar\lambda_Q}}F(-N, N + |\mathbb{m}_Q| + 1 + \frac{m\omega'_Q}{\lambda_Q\hbar}, |\mathbb{m}_Q| + 1; \sin(\Theta(x))).\end{aligned}$$

and the energy spectrum

$$E_{Q(N,\mathbb{m}_Q)} = \hbar\omega'_Q(2N + |\mathbb{m}_Q| + 1) + \frac{\lambda_Q\hbar^2}{2m}(2N + |\mathbb{m}_Q| + 1)^2, \tag{12}$$

which $\omega'_Q = \sqrt{\omega^2 + \frac{\hbar^2\lambda_Q^2}{4m^2}}$, N and $\mathbb{m}_Q$ are both integer number here.

C. The transformation from generalized CRS harmonic oscillator to radial Higgs model

Comparing the energy spectrum (8) and (12), if $\lambda_G = \lambda_Q = \lambda$ and $\mathbb{m}_G = \mathbb{m}_Q = \mathbb{m}$, it is obviously that they are exactly same. With the transformation

$$\Theta(x) = \text{arcsinh}(\sqrt{\lambda}x) = \Theta(x(r)) = \Upsilon(r) = \arctan(\sqrt{\lambda}r) \tag{13}$$

and separating the wave function (11)

$$\phi(x) = \phi(x(r)) = g(r)\psi(r), \quad g(r) = \left(-\sin^2(2\Upsilon(r))\right)^{-\frac{3}{4}}\sin^2(\Upsilon(r)), \tag{14}$$

we get the same differential equation as (6) and the same wave-function as (7).

Thus, we find the transformation relation here. If the Hamiltonian (1) with potential $V(x)$ can be solved exactly with the wave function $\phi(x)$, the radial part of Hamiltonian (4) with $\mathcal{V}(r)$ above also can be solved exactly with the following wave function

$$\psi(r) = \csc(\Upsilon(r))^2\left(-\sin(2\Upsilon(r))^2\right)^{\frac{3}{4}}\phi(x(r)), \tag{15}$$

which $V(x)$ and $\mathcal{V}(r)$ satisfies the relation

$$\mathcal{V}(r) = V(x(r)) + \frac{\lambda\hbar^2}{8m}\left(1 + (1 - 4\mathbb{m}_Q^2)\csc^2(\Upsilon(r))\right). \tag{16}$$

III. THE QUASI-EXACT MODEL IN TWO-DIMENSIONAL CURVED SPACE

For the transformation from generalized CRS harmonic oscillator to radial Higgs model, it can be easily found that the potential $\mathcal{V}(r)$ in two-dimensional Higgs model can only be solved exactly while the angular parameter $\mathbb{m}_G$ equals to the real number $\mathbb{m}_Q$. For $\mathbb{m}_Q \neq \mathbb{m}_G$ case, the 2-dimensional Higgs model is a quasi-exact model for angular part of this model can not be exactly solved.

Here, we would like to give some explicit examples, which satisfy $\lambda_G = \lambda_Q = \lambda$.

eg: (1) $X(x) = \cos(2\Theta(x)), \Theta(x) = \operatorname{arcsinh}(\sqrt{\lambda}x), \beta = 2\lambda(\mathbb{m}_Q + 1) + \frac{2m\omega'}{\hbar}, \gamma = 2\lambda\mathbb{m}_Q - \frac{2m\omega'}{\hbar}, A = -4\lambda, B = 0, C = \epsilon\left(\lambda(\mathbb{m}_Q^2 - 1) + \frac{2m\omega'}{\hbar}\mathbb{m}_Q\right), \omega' = \sqrt{\omega^2 + \frac{\hbar^2\lambda^2}{4m^2}}$. Thus, we have $\mathcal{V}(r) = \frac{1}{2}m\omega^2 r^2$. However, the wavefunction is

$$\Psi(r, \theta; N, \mathbb{m}_G, \mathbb{m}_Q) = e^{i\mathbb{m}_G\theta}\psi(r; N, \mathbb{m}_Q) \tag{17}$$

and

$$\psi\left(r; N, \mathbb{m}_Q\right) = r^{|\mathbb{m}_Q|}\left(\frac{1}{1+\lambda r^2}\right)^{\frac{|\mathbb{m}_Q|+2}{2}+\frac{m\omega'}{2\hbar\lambda}} F(-N, N+|\mathbb{m}_Q|+1+\frac{m\omega'}{\lambda\hbar}, |\mathbb{m}_Q|+1; \frac{\lambda r^2}{1+\lambda r^2})$$

From equation (17), it is obviously that this is a quasi-exact model.

eg: (2) $X(x) = x$, which means $A = \lambda, B = 0$. β is an arbitrary real numbers about parameter $\mathbb{m}_Q$. γ and C equals to 0. Thus, we have

$$\mathcal{V}(r) = \frac{\beta_{\mathbb{m}_Q}(\beta_{\mathbb{m}_Q} + \lambda)}{2\lambda} \tanh^2(\Upsilon(r)) + \frac{\lambda\hbar^2}{8m}\left(1 + (1 - 4\mathbb{m}_Q^2)\csc^2(\Upsilon(r))\right).$$

The ground state of wave function is

$$\Psi(r, \theta; 0, \mathbb{m}_G, \mathbb{m}_Q) = e^{i\mathbb{m}_G\theta}\psi(r; 0, \mathbb{m}_Q) \tag{18}$$

and

$$\psi\left(r; 0, \mathbb{m}_Q\right) = \csc(\Upsilon(r))^2\left(-\sin(2\Upsilon(r))^2\right)^{\frac{3}{4}} \operatorname{sech}\left(\Upsilon(r)\right)^{\frac{\beta_{\mathbb{m}_Q}}{\lambda}}.$$

From equation (18), it says that this quasi-exact model can be built by the transformation(13) .

IV. CONCLUSION

From the transformation(13), we can establish lots of quasi-exact models in two-dimensional curved space. This is a progress about quasi-exact theory and also a connection between quantum nonlinear harmonic oscillator (QNHO) theory and curved space model.

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